Abstract Operator Systems over the Cone of Positive Semidefinite Matrices

Project: Foundations of Free Algebraic Statistics

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Introduction

There are several important abstract operator systems with the convex cone of positive semidefinite matrices at the first level. Well-known are the operator systems of separable matrices, of positive semidefinite matrices, and of block positive matrices. In terms of maps, these are the operator systems of entanglement breaking, completely positive, and positive linear maps, respectively.

But there exist other interesting and less well-studied such operator systems, for example those of completely copositive maps, doubly completely positive maps, and decomposable maps. We investigate which of these systems is finitely generated, and which admits a finite-dimensional realization in the sense of the Choi-Effros Theorem.

Abstract Operator Systems

Let $\mathcal{V}$ denote a $\mathbb{C}$-vector space with involution $*$, $\mathcal{V}_h$ the $\mathbb{R}$-subspace of its Hermitian elements and for any $s \geq 1$ let $\mathrm{Mat}_s(\mathcal{V}) = \mathcal{V} \otimes_{\mathbb{C}} \mathrm{Mat}_s(\mathbb{C})$ be the vector space of $s \times s$-matrices with entries from $\mathcal{V}$.

Definition (e.g. [3, Chapter 13])

An *abstract operator system* $\mathcal{C}$ on $\mathcal{V}$ consists of a closed and salient convex cone $\mathcal{C}_s \subseteq \mathrm{Mat}_s(\mathcal{V})_h$ for each $s \geq 1$, and some $u \in \mathcal{C}_1 \subseteq \mathcal{V}_h$ such that

(i) $A \in \mathcal{C}_s, V \in \mathrm{Mat}_{s,t}(\mathbb{C}) \Rightarrow V^*AV \in \mathcal{C}_t$,
(ii) $u \otimes I_s$ is an order unit of $\mathcal{C}_s$ for all $s \geq 1$.

By the Choi–Effros Theorem [2] an abstract operator system $\mathcal{C}$ is *finite-dimensional realizable* if it constitutes a free spectrahedron, i. e. there exist hermitian matrices $T_1, \ldots, T_d$ such that

$$\mathcal{C}_s = \left\{(A_1, \ldots, A_d) \in \mathrm{Her}_s(\mathbb{C})^d \;\middle|\; T_1 \otimes A_1 + \cdots + T_d \otimes A_d \succcurlyeq 0\right\}.$$

Note that in the above situation $\mathcal{C}_1$ coinces with the notion of a (classical) spectrahedron.
An operator system is called *finitely generated* if there exists $A \in \mathcal{C}_s$ such that each element from $\mathcal{C}$ is of the form

$$\sum_i V_i^* A V_i = V^*(A \oplus \cdots \oplus A)V$$

for complex matrices V, V_i (of the correct size), which is equivalent to being the smallest operator system containing A.

Operator Systems Over the Cone of Positive Matrices

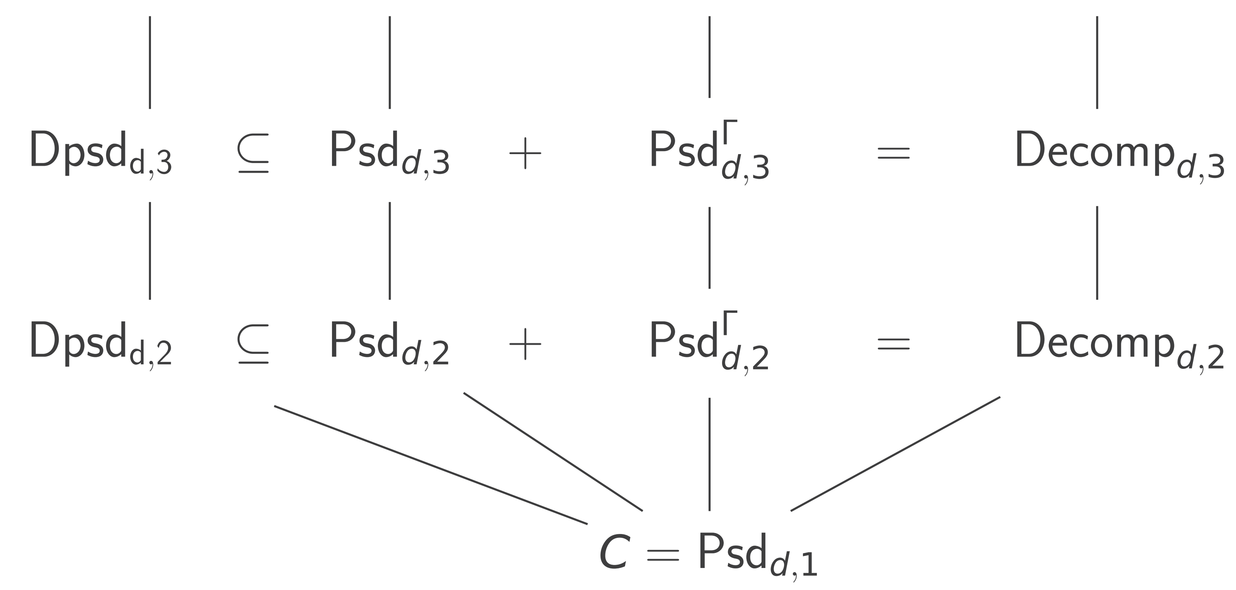

Figure 1: Schematic representation of the relationships between the cones discussed below.

In the following let $C \subseteq \mathrm{Her}_d(\mathbb{C})$ be the cone of positive semidefinite complex $d \times d$ matrices. We are interested in operator systems with C at level one.
Two such systems are the operator system of positive semidefinite matrices $\mathrm{Psd}_d = (\mathrm{Psd}_{d,s})_{s \in \mathbb{N}}$, defined by

$$\mathrm{Psd}_{d,s} := \left\{ \sum_{i=1}^n A_i \otimes B_i \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) \;\middle|\; n \in \mathbb{N}, \sum_{i=1}^n A_i \otimes B_i \succcurlyeq 0 \right\}$$

and the operator system of matrices with positive partial transpose $\mathrm{Psd}^{\Gamma}_d = (\mathrm{Psd}^{\Gamma}_{d,s})_{s \in \mathbb{N}}$, where

$$\mathrm{Psd}^{\Gamma}_{d,s} := \left\{ \sum_{i=1}^n A_i \otimes B_i \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) \;\middle|\; n \in \mathbb{N}, \sum_{i=1}^n A_i^T \otimes B_i \succcurlyeq 0 \right\}$$

Both of the above operator systems are finitely generated as well as finite-dimensional realizable. We obtain the operator system of *doubly positive matrices* $\mathrm{Dpsd}_d = (\mathrm{Dpsd}_{d,s})_{s \in \mathbb{N}}$ as the level-wise intersection of the systems Psd_d and Psd^{Γ}_d, i.e. by

$$\mathrm{Dpsd}_{d,s} := \mathrm{Psd}_{d,s} \cap \mathrm{Psd}^{\Gamma}_{d,s}$$

and the operator system of *decomposable matrices* $\mathrm{Decomp}_d = (\mathrm{Decomp}_{d,s})_{s \in \mathbb{N}}$ as the level-wise Minkowski sum of the systems Psd_d and Psd^{Γ}_d, i.e. by

$$\mathrm{Decomp}_{d,s} := \left\{ X + Y \,\middle|\, X \in \mathrm{Psd}_{d,s}, Y \in \mathrm{Psd}^{\Gamma}_{d,s} \right\}.$$

Results

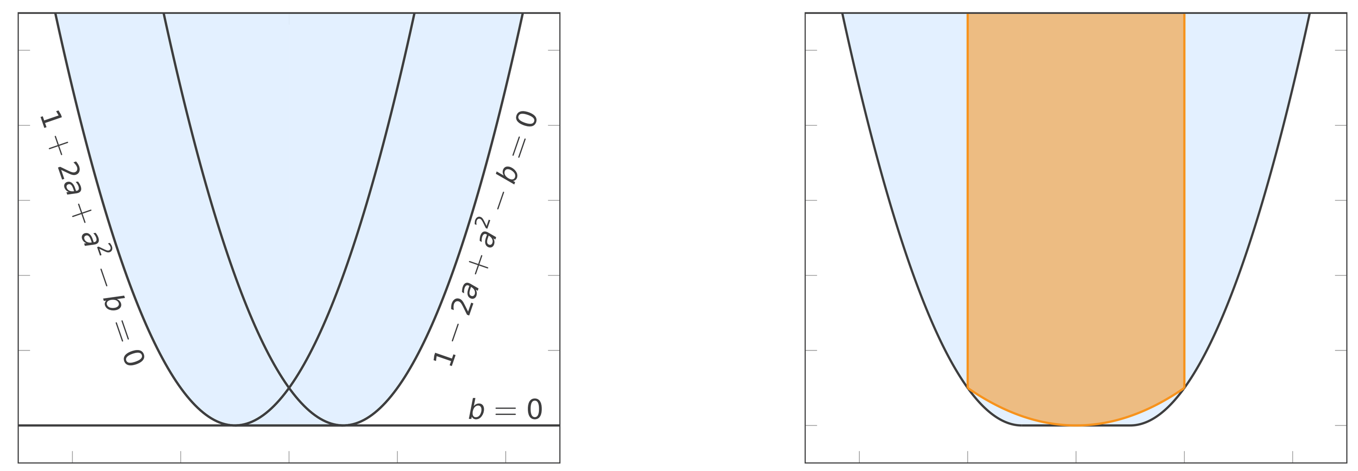

Figure 2: Left: Our two-dimensional subspace and its algebraic boundary. Right: The sets $\mathrm{Psd}_{2,2} = \mathrm{Psd}^{\Gamma}_{2,2}$ (orange) and $\mathrm{Decomp}_{2,2}$ (blue) in our two-dimensional subspace.

Theorem ([1])

(i) *The convex cone* $\mathrm{Decomp}_{2,2}$ *has a non-exposed face, and is not basic closed semialgebraic. In particular it is not a (classical) spectrahedron.*
(ii) *For* $d \geq 2$*, the operator system* Decomp_d *of decomposable matrices does not admit a finite-dimensional realization, and the operator system* Dpsd_d *of doubly positive matrices is not finitely generated.*

Remark:

- We proofed our result by showing that $\mathrm{Decomp}_{2,2}$ is not a spectrahedron. We exhibited a two-dimensional subspace on which $\mathrm{Decomp}_{2,2}$ fails to fulfill two necessary conditions for having a linear matrix inequality definition, see Figure 2.
- Our result shows that intersections of two finitely generated operator systems need not be finitely generated, and Minkowski sums of two operator systems with a finite-dimensional realization need not have a finite-dimensional realization.

References:

[1] Martin Berger and Tim Netzer, *Abstract operator systems over the cone of positive semidefinite matrices* (2021), available at 2109.14453.
[2] M.-D. Choi and E. G. Effros, *Injectivity and operator spaces*, Journal of Functional Analysis **24** (1977), no. 2, 156–209.
[3] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2003.

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