Thick points of random walk and Gaussian multiplicative chaos

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Introduction

Thick points of planar Brownian motion/random walk are points that have been visited unusually often by the trajectory. The study of these points has a long history going back to the famous conjecture of Erdős and Taylor [4] on the leading order of the number of times a planar simple random walk visits the most visited site during the first n steps. This conjecture has been solved forty years later in [3] but these thick points remained so far very mysterious. The purpose of this PhD thesis is to shed light on these exceptional points taking in particular advantage of recent developments on Gaussian multiplicative chaos.

Brownian multiplicative chaos

In this section we explain how to build random Borel measures supported on thick points of planar Brownian motion. These measures can be formally defined as

$$m^\gamma(dx) := e^{\gamma \sqrt{L_x}} dx$$

where $\gamma \in [0,2]$ is a parameter, dx is the Lebesgue measure and L_x denotes the local time of planar Brownian motion (suitably stopped) at x. This definition is only formal since L_x is not defined pointwise. We will see that we can make sense of such a measure by going though an approximation procedure. We will first focus on the subcritical regime where $\gamma < 2$ and we will then move on to the critical regime where $\gamma = 2$. This section is based on [5] and [7].

Subcritical regime

Let $\mathbb{P}_x$ be the law under which $(B_t)_{t \geq 0}$ is a planar Brownian motion starting from $x \in \mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be an open bounded simply connected domain, $x_0 \in D$ be a starting point and τ be the first exit time of D. For all $x \in \mathbb{R}^2, t > 0, \varepsilon > 0$, define the local time $L_{x,\varepsilon}(t)$ of $(|B_s - x|, s \geq 0)$ at ε up to time t (here $|\cdot|$ stands for the Euclidean norm):

$$L_{x,\varepsilon}(t) := \lim_{r \to 0^+} \frac{1}{2r} \int_0^t \mathbf{1}_{\{\varepsilon - r \leq |B_s - x| \leq \varepsilon + r\}} ds. \tag{1}$$

In [5, Proposition 1.1], we show that we can make sense of the local times $L_{x,\varepsilon}(\tau)$ simultaneously for all x and ε with the convention that $L_{x,\varepsilon}(\tau) = 0$ if the circle $\partial D(x,\varepsilon)$ is not entirely included in D. We can thus define for any thickness parameter $\gamma \in (0,2]$ and any Borel set A,

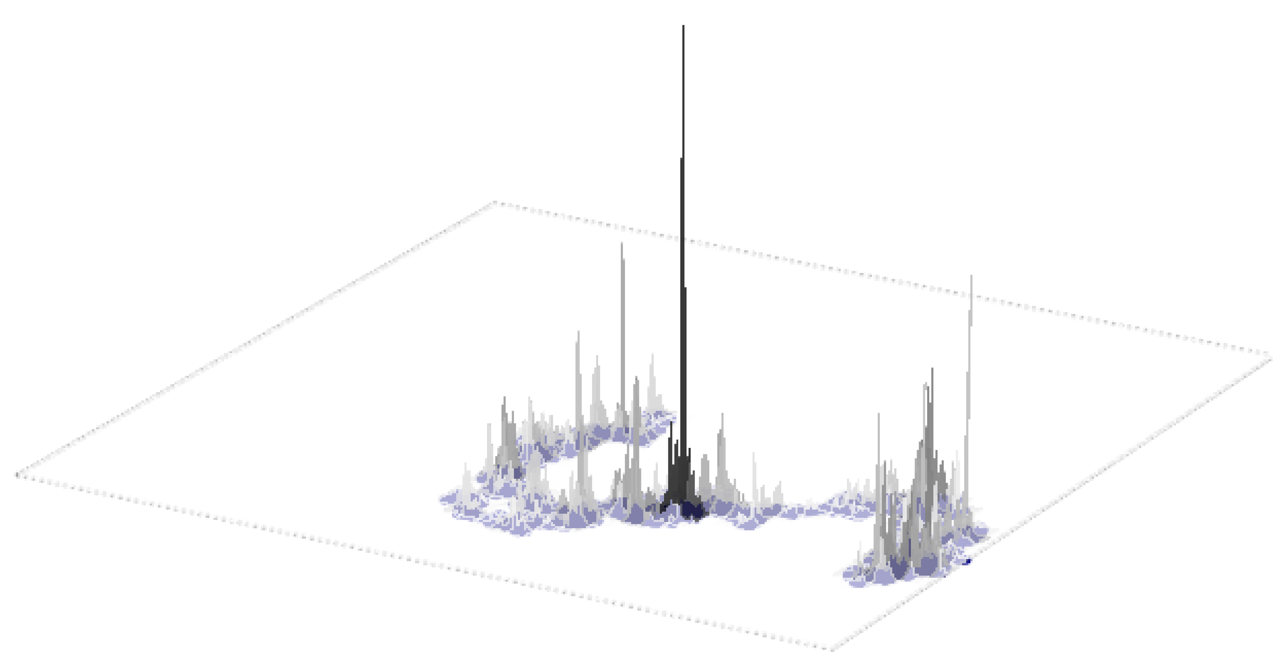
$$m_\varepsilon^\gamma(A) := \sqrt{|\log \varepsilon|} \varepsilon^{\gamma^2/2} \int_A e^{\gamma \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} dx. \tag{2}$$

The main theorem of [5] is:

Theorem 1 Let $\gamma \in (0,2)$. The sequence of random measures m_ε^γ converges as $\varepsilon \to 0$ in probability for the topology of weak convergence on D towards a Borel measure m^γ called Brownian multiplicative chaos.

See [1] for a different construction of the subcritical Brownian multiplicative chaos, as well as [2] for partial results. See also [6] for more properties on these measures.

The figure below shows a simulation of subcritical Brownian multiplicative chaos with parameter $\gamma = 1$. The underlying Brownian trajectory is depicted in blue. The domain D is a square and the starting point is its centre.



Critical regime

$\gamma = 2$ appears to be a critical point. Indeed, at this point the subcritical normalisation yields a vanishing measure:

Proposition 1 (Proposition 1.1 of [7]) $m_\varepsilon^{\gamma=2}(D)$ converges in $\mathbb{P}_{x_0}$-probability to zero.

To obtain a non-trivial object we thus need to renormalise the measure slightly differently. Firstly, we consider the Seneta–Heyde normalisation: for all Borel set A, define

$$m_\varepsilon(A) := \sqrt{|\log \varepsilon|} m_\varepsilon^{\gamma=2}(A) = |\log \varepsilon| \varepsilon^2 \int_A e^{2\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} dx. \tag{3}$$

Secondly, we consider the derivative martingale normalisation which formally corresponds to (minus) the derivative of m_ε^γ with respect to γ evaluated at $\gamma = 2$: for all Borel set A, define

$$\mu_\varepsilon(A) := -\frac{\mathrm{d} m_\varepsilon^\gamma(A)}{\mathrm{d}\gamma}\Big|_{\gamma=2} = \sqrt{|\log \varepsilon|} \varepsilon^2 \int_A \left(-\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)} + 2 \log \frac{1}{\varepsilon} \right) e^{2\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} dx. \tag{4}$$

Theorem 2 (Theorem 1.1 of [7]) The sequences of random positive measures $(m_\varepsilon)_{\varepsilon > 0}$ and random signed measures $(\mu_\varepsilon)_{\varepsilon > 0}$ converge in $\mathbb{P}_{x_0}$-probability for the topology of weak convergence towards random Borel measures m and μ. Moreover, the limiting measures satisfy:

1. $m = \sqrt{\frac{2}{\pi}} \mu$ $\mathbb{P}_{x_0}$-a.s. In particular, μ is a random positive measure.
2. Nondegeneracy: $\mu(D) \in (0,\infty)$ $\mathbb{P}_{x_0}$-a.s.
3. First moment: $\mathbb{E}_{x_0}[\mu(D)] = \infty$.
4. Nonatomicity: $\mathbb{P}_{x_0}$-a.s. simultaneously for all $x \in D$, $\mu(\{x\}) = 0$.

Critical Brownian multiplicative chaos can also be constructed as a limit of subcritical measures:

Theorem 3 (Theorem 1.2 of [7]) $(2-\gamma)^{-1} m^\gamma$ converges towards 2μ as $\gamma \to 2^-$ in probability for the topology of weak convergence of measures.

Thick points of planar random walk

This section is dedicated to thick points of planar random walk. We will see that Brownian multiplicative chaos measures defined in the previous section describe the scaling limit of these thick points.

Let $D \subset \mathbb{R}^2$ be an open bounded simply connected domain and let $x_0 \in D$ be a starting point. Let N be a large integer and let

$$D_N := \{\lfloor Nx \rfloor : x \in D\} \subset \mathbb{Z}^2$$

be a discrete approximation of D. Let $(X_t)_{t \geq 0}$ be a continuous time simple random walk on $\mathbb{Z}^2$ with jump rate one (at every vertex, it waits an exponential time with parameter one before jumping) and define its exit time τ_N of D_N and local times: for $x \in \mathbb{Z}^2$,

$$\ell_x^{\tau_N} := \int_0^{\tau_N} \mathbf{1}_{\{X_s = x\}} ds.$$

For $x \in \mathbb{C}$, we will denote $\mathbb{P}_x^N$ the probability measure associated to the walk $(X_t, t \leq \tau_N)$ starting at $X_0 = \lfloor x \rfloor$.

Let $a \in (0,2)$ be a parameter measuring the thickness level, $g := 2/\pi$. We encode the set of thick points in a point measure m_N^a by setting for all Borel sets $A \subset \mathbb{C}$,

$$m_N^a(A) := \frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in A\}} \mathbf{1}_{\{\ell_x^{\tau_N} \geq g a \log^2 N\}} \text{ under } \mathbb{P}_{N x_0}^N. \tag{5}$$

Theorem 4 (Theorem 1.1 of [6]) For all $a \in (0,2)$, the sequence m_N^a, $N \geq 1$, converges weakly for the topology of weak convergence on D. Moreover, there exists a universal constant c_0 such that the limiting measure has the same distribution as $e^{c_0 a/g} m^{\gamma = \sqrt{2a}}$ built in Theorem 1.

At criticality, we expect the following picture. Define for all $N \geq 1$, define the measure μ_N on $\mathbb{R}^2 \times \mathbb{R}$ by setting for all Borel sets $A \subset \mathbb{R}^2$ and $T \subset \mathbb{R}$,

$$\mu_N(A \times T) := \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in A\}} \mathbf{1}_{\left\{\sqrt{\ell_x^N} - 2\pi^{-1/2} \log N + \pi^{-1/2} \log\log N \in T\right\}}.$$

Conjecture 1 There exist constants $c_1, c_2 > 0$ such that $(\mu_N, N \geq 1)$ converges in distribution for the topology of vague convergence on $\mathbb{R}^2 \times (\mathbb{R} \cup \{+\infty\})$ towards

$$\mathrm{PPP}(c_1 \mu \otimes c_2 e^{-c_2 t} dt)$$

where μ is the critical Brownian multiplicative chaos in the domain $[-1,1]^2$ with the origin as a starting point. In particular, for all $t \in \mathbb{R}$,

$$\mathbb{P}\left(\sup_{x \in \mathbb{Z}^2} \sqrt{\ell_x^N} \leq \frac{2}{\sqrt{\pi}} \log N - \frac{1}{\sqrt{\pi}} \log\log N + t \right) \xrightarrow[N \to \infty]{} \mathbb{E}\left[\exp\left(-c_1 \mu([-1,1]^2) e^{-c_2 t} \right) \right].$$

This conjecture shall be addressed in forthcoming research.

References

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