Forcing theory and combinatorics of the real line

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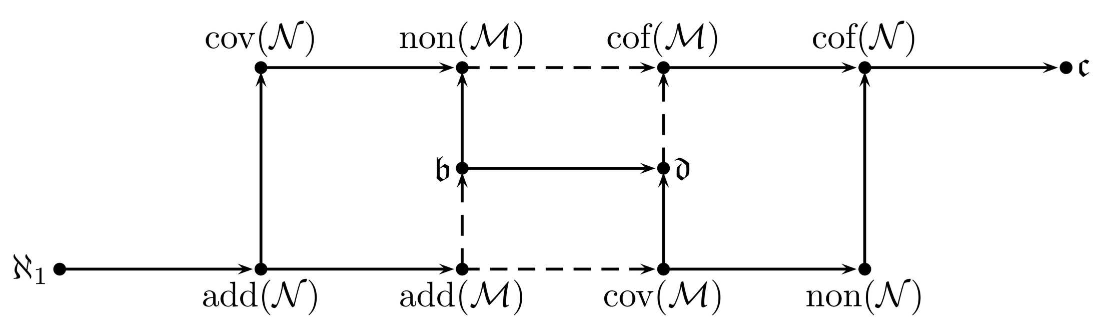
Introduction

The history of the cardinality of the continuum was begun by George Cantor with his classical result which says that $2^{\aleph_0} > \aleph_0$, where $\aleph_0$ denote the size of the set natural numbers and $2^{\aleph_0}$, the size of *the continuum*, is the size of the set of real numbers. This is the first theorem cardinal characteristic of the continuum. Many combinatorial properties of the real line can be associated with cardinal numbers known as *cardinal characteristics of the continuum*. As an example: for an ideal $\mathcal{I}$ of subsets of $\mathbb{R}$ the cardinal characteristics $\mathrm{add}(\mathcal{I})$, $\mathrm{cov}(\mathcal{I})$, $\mathrm{non}(\mathcal{I})$ are defined as the answers to the questions:

- How many ideal sets do you need to *add up* to get a non-ideal set? This is $\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} \notin \mathcal{I}\}$ the *additivity of* $\mathcal{I}$.
- How many ideal sets do you need to *cover* all of $\mathbb{R}$? This is $\mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} = \mathbb{R}\}$ the *covering of* $\mathcal{I}$.
- How many points of $\mathbb{R}$ do Theoremyou need to get a *non*-ideal set? The goal This is $\mathrm{non}(\mathcal{I}) := \{|A| : F \subseteq \mathbb{R}, A \notin \mathcal{I}\}$ the *uniformity of* $\mathcal{I}$.

We also define the *cofinality* of $\mathcal{I}$, denoted $\mathrm{cof}(\mathcal{I})$, as the smallest cardinality of a set that is cofinal in the partial order $(\mathcal{I}, \subseteq)$.

A classical example is Cichoń's diagram (see the figure below), which illustrates the cardinals associated with $\mathcal{N}$, the ideal of Lebesgue null sets, and with $\mathcal{M}$, the ideal of meager (or first category) sets, as well as the numbers $\mathfrak{b}$ and $\mathfrak{d}$ (the unbounding number and the dominating number, or equivalently the additivity number and covering number of the σ-ideal generated by the compact sets of irrationals), and the cardinal numbers $\aleph_1$ and $\mathfrak{c} := 2^{\aleph_0}$.



The ideal $\mathcal{SN}$ of *strong measure zero* subsets of the reals started to receive a lot of attention since it was discovered that the *Borel's conjecture*, which states that every strong measure zero set is countable, cannot be proven nor refuted in ZFC (CH implies that it is false and, on the other hand, R. Laver [14] proved its consistency with ZFC by forcing). Afterwards, the cardinal invariants associated with $\mathcal{SN}$ have been interesting objects of research, in particular when related to the cardinals in Cichoń's diagram. In this context, the most impressive result so far is the consistency of $\mathfrak{c} < \mathrm{cof}(\mathcal{SN})$ with ZFC, which was proved by Yorioka [16]. To do this, he [16] gave a very useful characterization of $\mathcal{SN}$ in terms of σ-ideals $\mathcal{I}_f$ parametrized by increasing functions $f \in \omega^\omega$, which are known as *Yorioka ideals*. Concretely, $\mathcal{SN} = \bigcap\{\mathcal{I}_f : f \in \omega^\omega \text{ increasing}\}$ and $\mathcal{I}_f \subseteq \mathcal{N}$.

Let $b = \langle b(n) : n < \omega\rangle$ be a sequence of non-empty sets and let $h \in \omega^\omega$. Denote:

$$\prod b := \prod_{i<\omega} b(i) \text{ and } \mathcal{S}(b,h) = \prod_{n<\omega} [b(n)]^{\leq h(n)}.$$

Define the cardinals numbers $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$(called *localization cardinals*) and $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$(called *anti-localization cardinals*) as follows: Define the following cardinal characteristics.

$$\mathfrak{d}^{\mathrm{Lc}}_{b,h} := \min\left\{|D| : D \subseteq \mathcal{S}(b,h),\ \forall x \in \prod b\, \exists \varphi \in D\, (x \in^* \varphi)\right\},$$

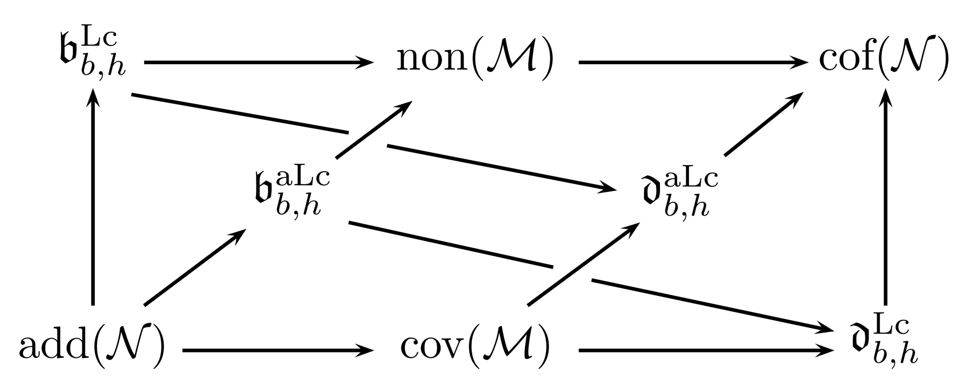
$$\mathfrak{b}^{\mathrm{Lc}}_{b,h} := \min\left\{|F| : F \subseteq \prod b,\ \neg\exists \varphi \in \mathcal{S}(b,h)\, \forall x \in F\, (x \in^* \varphi)\right\},$$

$$\mathfrak{b}^{\mathrm{aLc}}_{b,h} := \min\left\{|F| : F \subseteq \mathcal{S}(b,h),\ \neg\exists x \in \prod b\, \forall \varphi \in F\, (x \in^\infty \varphi)\right\},$$

$$\mathfrak{d}^{\mathrm{aLc}}_{b,h} := \min\left\{|D| : D \subseteq \prod b,\ \forall \varphi \in \mathcal{S}(b,h)\, \exists x \in D\, (x \notin^\infty \varphi)\right\}.$$

The localisation and anti-localisation cardinals have appeared in many contexts. The following are well-known characterisations: The localisation and anti-localisation cardinals are a generalisation of the cardinals used in Bartoszyński's characterisations $\mathrm{add}(\mathcal{N}) = \mathfrak{b}^{\mathrm{Lc}}_{\omega,h}$ and $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}^{\mathrm{Lc}}_{\omega,h}$ when h diverges to infinity (here ω is interpreted as the constant sequence ω), $\mathrm{non}(\mathcal{M}) = \mathfrak{b}^{\mathrm{aLc}}_{\omega,h}$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}^{\mathrm{aLc}}_{\omega,h}$ when $h \geq^* 1$, (see e.g Bartoszyński [1] and Bartoszyński [2], Miller [15].

The figure below illustrates the provable inequalities among localisation (when h diverges to infinity), anti-localization cardinals (when $\frac{h(n)}{b(n)}$ converges to 0), and the cardinals in Cichoń's diagram (see [8] for a summary). In addition, as hinted in [9] and proved in [13, Lemma 2.3]: if $\sum_{n<\omega} \frac{h(n)}{b(n)}$ converges then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{N})$; and if $\sum_{n<\omega} \frac{h(n)}{b(n)}$ diverges then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{N})$.



Consistency results about many different values for localization and anti-localization cardinals have been investigated. Some of the earliest results is due to Goldstern and Shelah [9] who constructed a poset using creatures to force that $\aleph_1$-many cardinals of the form $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ are pairwise different, which solves a question of Blass [3] result that was later improved by Kellner [10] showing the consistency of $\mathfrak{c}$-many pairwise different cardinals of the same form, and Kellner and Shelah [11, 12] included similar consistency results for the form $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$. Recently, Klausner and Mejía [13] using product of creature proved that $\aleph_1$-many cardinals of the form $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ are pairwise differents.

Objectives and scientific problems

The aim of this Ph.D. thesis is to apply and develop new forcing techniques to obtain models where several cardinal characteristics are pairwise different, in particular those that we discussed up to this point. In addition, we aim to force many (even more continuum many) different values of cardinal characteristics that are parametrised by reals, in particular localization, and anti-localization cardinals, and cardinal characteristics associated with $\mathcal{I}_f$. This naturally leads to the following main problems.

Main Problem A. *Produce models by forcing where the four cardinal characteristics associated with an ideal $\mathcal{I}$ are pairwise different. Particular emphasis will be given to $\mathcal{N}$, $\mathcal{M}$, $\mathcal{SN}$ and $\mathcal{I}_f$.*

Note that, for an ideal $\mathcal{I}$, there can be at most two cases for Main problem A, namely

$(A1)_{\mathcal{I}}$ $\mathrm{add}(\mathcal{I}) < \mathrm{cov}(\mathcal{I}) < \mathrm{non}(\mathcal{I}) < \mathrm{cof}(\mathcal{I})$, and

$(A2)_{\mathcal{I}}$ $\mathrm{add}(\mathcal{I}) < \mathrm{non}(\mathcal{I}) < \mathrm{cov}(\mathcal{I}) < \mathrm{cof}(\mathcal{I})$.

Main Problem B. *Is it consistent with ZFC that $\mathfrak{c}$-many cardinal characteristics of the form $\mathrm{cov}(\mathcal{I}_f)$ are pairwise different? The same question is asked for cardinals of the type $\mathrm{non}(\mathcal{I}_f)$, $\mathrm{cof}(\mathcal{I}_f)$ and $\mathrm{add}(\mathcal{I}_f)$ and localisation, anti-localisation cardinals instead.*

Main results

The main results of this Ph.D. thesis were obtained in joint work with J. Brendle, L. Klausner and D. Mejía, whose outcome are the scientific papers [4, 7, 5, 6]. Below, the following theorem solved some problems from Main problem A.

Main Theorem 1. *It is consistent with ZFC each one of the following statements*

1. ([4]) $(A2)_{\mathcal{M}}$.
2. ([4]) $(A1)_{\mathcal{I}_f}$ for any $f : \mathbb{N} \to \mathbb{N}$.
3. ([5]) $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{I}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.
4. ([7]) $\mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{I}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.

The next theorem solved several questions from Main Problem B. Concretely, we prove the following theorem.

Main Theorem 2 ([6])**.** *Assume CH. Then there is an $\aleph_2$-cc ω^ω-bounding proper poset forcing that there are continuum many pairwise different cardinal characteristics of each one of the 6 types $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$, $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$, $\mathrm{non}(\mathcal{I}_f)$ and $\mathrm{cof}(\mathcal{I}_f)$.*

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