Rotation Invariant Diagrammatic Bases for Invariant Spaces

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Project Goal

The goal of the project is to give simple pictorial descriptions of bases of algebraically interesting vector spaces (invariant spaces for Lie Groups) in terms of chord diagrams.
There is a notion of *rotation* on the elements of these spaces. We are interested in bases that are closed under rotation and an "easy" description of this operation.

Algebraic Background

Let G be a simple Lie Group and let U be a representation. G acts on $U^{\otimes n}$ diagonally, i.e.

$$g \cdot (u_1 \otimes \ldots \otimes u_n) = (g \cdot u_1) \otimes \ldots \otimes (g \cdot u_n).$$

We want to study different bases of the invariant space

$$(U^{\otimes n})^G = \{u \in U^{\otimes n} \mid g \cdot u = u \, \forall g \in G\}.$$

The symmetric group $\mathfrak{S}_n$ acts on $U^{\otimes n}$ by permuting tensor positions and the action of the long cycle of the symmetric group can be regarded as an abstraction of *rotation*.

Describing a basis	rotation
messy	easy

Combinatorial Model

A basis of $(U^{\otimes n})^G$ is indexed by certain sequences of vectors ($\emptyset = \mu^0, \mu^1, \ldots, \mu^n = \mu$), which are called *tableaux*.

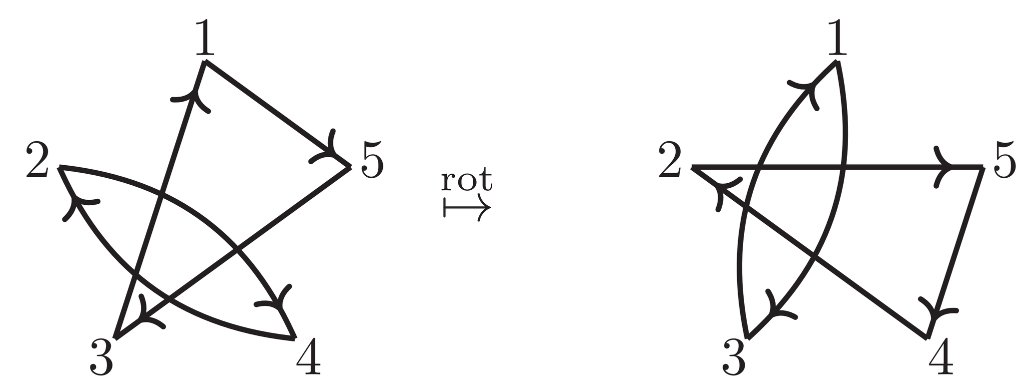
The action of the long cycle corresponds to *promotion*, which is defined recursively via the local rule $\mu = \mathrm{dom}_W(\kappa + \nu - \lambda)$, where $\kappa, \lambda, \mu, \nu$ are four corners of a cell in the following scheme:

Describing a basis	rotation
easy	messy

Chord Diagrams

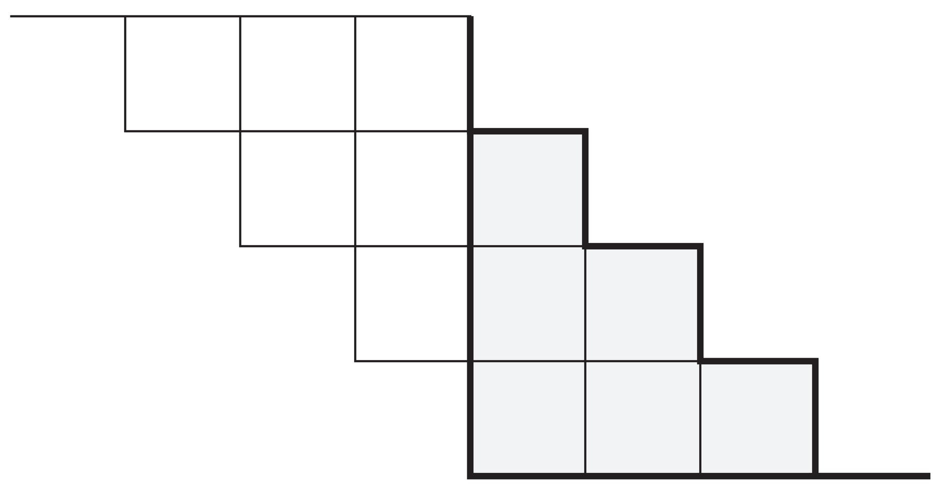
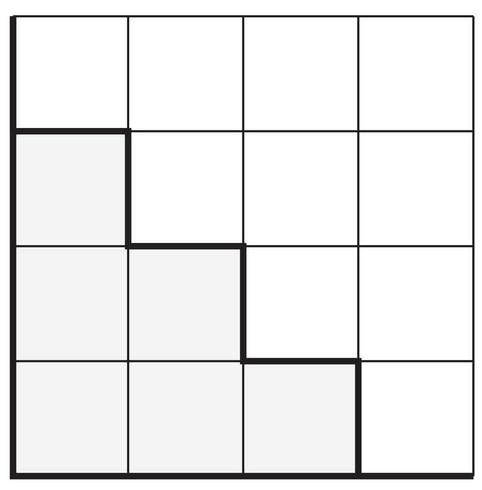
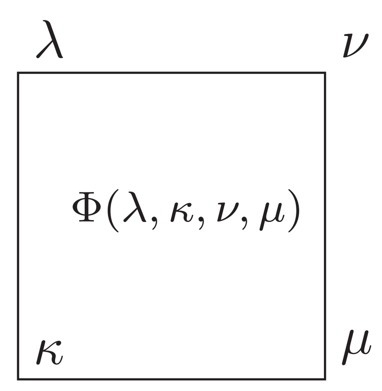
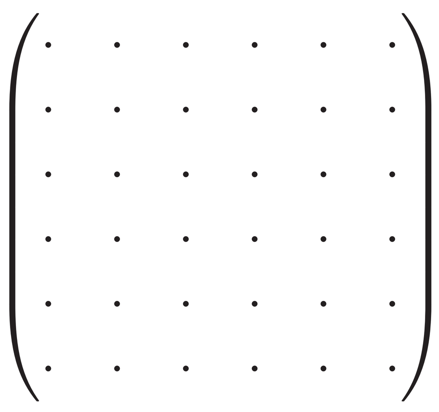
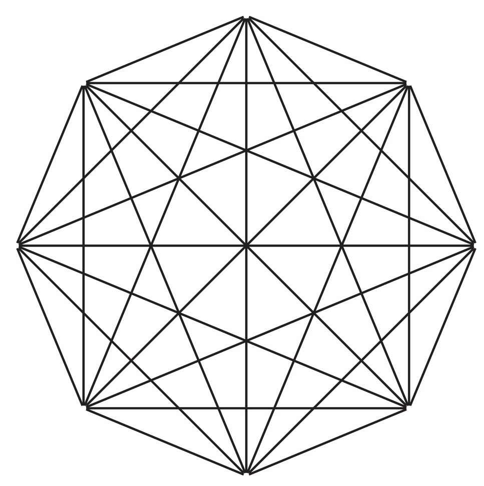
A *chord diagram of size n* is a graph with vertices $\{1, 2, \ldots, n\}$. It can be directed or undirected. We usually depict the vertices in a cycle.

The *rotation* of a chord diagram is obtained by rotating all the edges in clockwise direction.



Describing a basis	rotation
???	easy

Structural Approach

1. Calculate promotion over and over again using the calculation scheme	**2.** Cut and glue the scheme to obtain a square	**3.** Fill all cells according to a function Φ with integers	**4.** Interpret the filled square as adjacency matrix of a graph	**5.** Construct the chord diagram from the adjacency matrix.
				

Let $\mathcal{T}$ be a tableau and denote with $\mathcal{G}(\mathcal{T})$ the image of this construction. Then $\mathcal{G}(\mathrm{pr}\,\mathcal{T}) = \mathrm{rot}\,\mathcal{G}(\mathcal{T})$ and we get a rotation invariant set of chord diagrams.

Describing a basis in terms of chord diagrams reduces to:

Fix a group G and representation U. For this representation

- define a filling rule Φ,
- such that $\mathcal{G}$ is injective
- and give a direct description of all chord diagrams obtained by $\mathcal{G}$.

Objectives

Solve this problem for the representations:

Type A_r Vector representation.
Type B_r Spin and vector representation.
Type C_r Vector representation.
Type D_r Vector representation.

Result and Example for Type B_r, Spin Representation

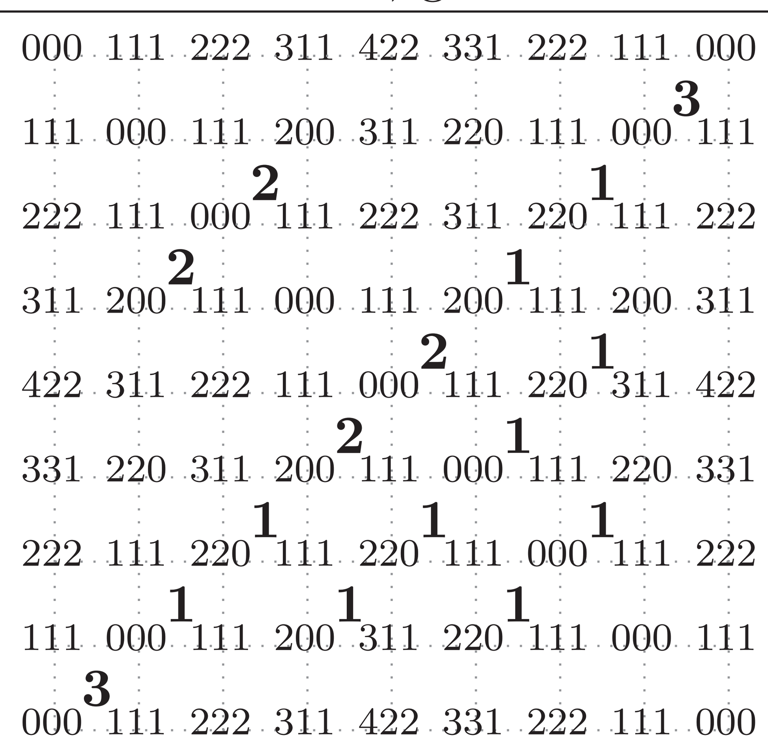
For this representation dom_W sorts the absolute values of coordinates of the vector and we set $\Phi(\lambda, \kappa, \nu, \mu) = \#$ of negative entries in $\kappa + \nu - \lambda$.

E.g. $\mathcal{T} = (000, 111, 222, 311, 422, 331, 222, 111, 000)$.

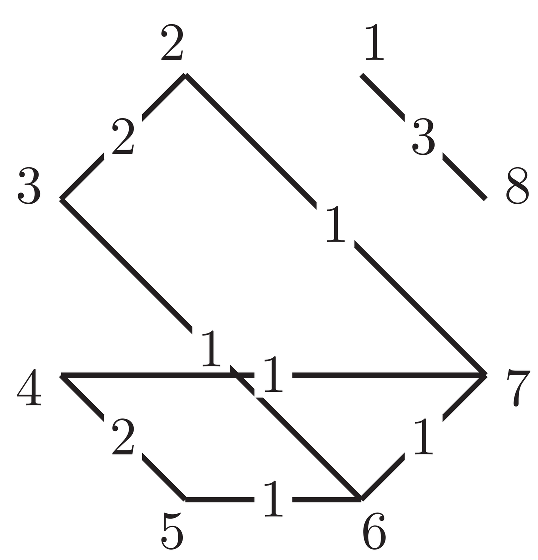
1. Promotion orbit

000 111 222 311 422 331 222 111 000
000 111 200 311 220 111 000 111 000
000 111 222 311 220 111 222 111 000
000 111 200 111 200 311 200 111 000
000 111 220 311 422 311 222 111 000
000 111 220 331 220 311 200 111 000
000 111 222 111 220 111 220 111 000
000 111 000 111 200 311 220 111 000
000 111 222 311 422 331 222 111 000

2. & 3. Cut, glue and fill



4. & 5. Construct the diagram



With this filling rule $\mathcal{G}$ is injective. Further, we found various alternative constructions to obtain the chord diagrams from $\mathcal{T}$ without calculating the promotion orbit. A direct description of all chord diagrams obtained by $\mathcal{G}$ is still missing.