Towards an axiomatization of the set theoretic multiverse

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Gödel's Incompleteness Theorems

There are problems in mathematics that do not have an answer in our current framework (Zermelo-Fraenkel set theory with Choice, ZFC), instantiated in the *cumulative hierarchy* V

e.g.

- **Continuum Hypthesis** (CH): $2^{\aleph_0} = \aleph_1$;
- **Generalized Continuum Hypothesis** (GCH): $2^{\aleph_n} = \aleph_{n+1}$;
- If a set A has fewer elements than B, then A also has fewer subsets than B;

Our current framework is thus *incomplete*, and we need to complete it in some way, so we can give an answer to some of those problems.

e.g.

By adding new axioms to ZFC, or new sets to V.

Is there a unique way to complete ZFC?

Yes

Universism

There is only one unique mathematical universe, V, and all of mathematics is carried out inside it. If it seems that there is something outside V, it is only an illusion.

The universe of set theory V is a *cumulative hierarchy*: hierarchy because all sets are organised in levels, and cumulative because from one level to the next one we add all new possible sets that can be formed using the set creation tool powerset.

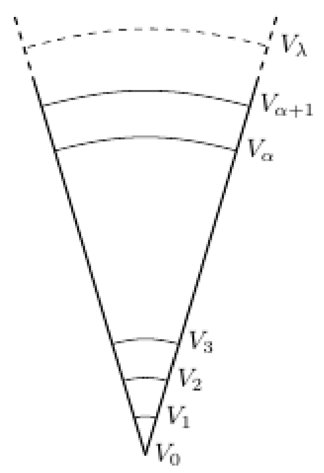

Figure 1: The cumulative hierarchy

- $V_0 = \emptyset$;
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$;
- $V_\lambda = \bigcup V_\alpha$ (for all $\alpha < \lambda$, where λ is a limit ordinal);

The following reasons are usually put forward as a defence of universism:

- categoricity;
- the toy model approach to forcing;
- overly complex mathematical characterization of the multiverse conceptions;
- pre-mathematical intuitions.

Toy model approach: when we use forcing to produce new models (universes), we are not actually doing it, but we are simulating everything inside of V.

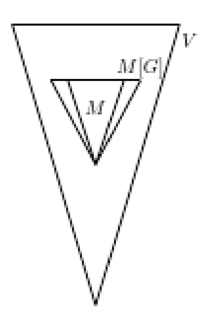

Figure 2: A representation of the toy model approach to forcing

Counterpoint: While enticing, the toy model approach to forcing does not really explain current set theoretic practice. Moreover, it also restricts the number of available theorems.

Much better (and closer to our current practice and intuition) is the *natural approach* to forcing. According to this approach, when using forcing we are applying it to the *whole* universe V, thus producing an extension $V[G]$ for it.

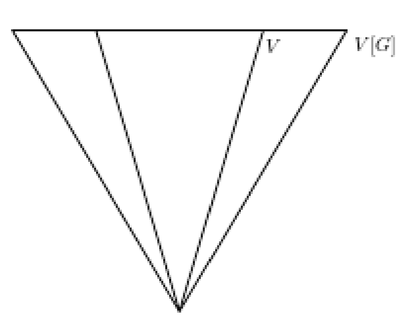

Figure 3: A representation of the natural interpretation to forcing

Categoricity: every model (=universe) of second order set theory (ZFC_2) is isomorphic (="equal"), so we are actually dealing with a single universe and not a multiverse.

Counterpoint: Such categoricity results need very strong assumptions, that are not easily justifiable (e.g. the passage to the second order, Martin's Uniqueness Postulate and McGee's Urelements Axiom).

- **Mathematical characterisation**: Classic set theory (ZFC) is axiomatized in a very simple and elegant way, while the multiverse conceptions are very complex to characterised and need an already developed and advanced knowledge of set theory;
- **Pre-mathematical intuition**: All of our pre-mathematical intuitions about set and membership are vindicated by ZFC and V, so there is no need to depart from them.

Counterpoints:

- **Mathematical characterisation**: This is a point more against certain particular multiverse conceptions than against the whole multiverse position. While some of the mathematical characterisation of it are indeed very complex, the multiverse can also be axiomatized in the same elegant way of ZFC;
- **Pre-mathematical intuition**: Again, this objection is aimed against some particular multiverse. It is possible to define such multiverses in which our pre-mathematical intuitions of set and membership are vindicated just like in ZFC and V.

For these reasons, a multiverse approach to set theory is much more tenable than a defence of universism and the Single Universe.

No

Multiversism

There is more than one set theoretic (mathematical) universe. Each one of these universes is equally legitimate, and it is produced mainly (but not only) by the use of forcing. These universes are then linked together to form a *multiverse*.

There are various reasons in favour of such a conception:

- Better explanation of incompleteness;
- Agreement with current set theoretic (mathematical) practice;
- Settlement of independent questions;
- Maximization of available theorem types;
- Conservation of previous results.

There are several different conceptions of the mathematical and philosophical characterisation of the set theoretic multiverse

Radical Multiverse: Every conceivable model of set theory is part of this multiverse, without any restriction on how it was produced (so we have universes produced by any type of forcing from any starting universe, and for all of these all the possible inner models, etc.). This multiverse axiomatization is satisfied by the collection of all countable computably saturated models of ZFC.

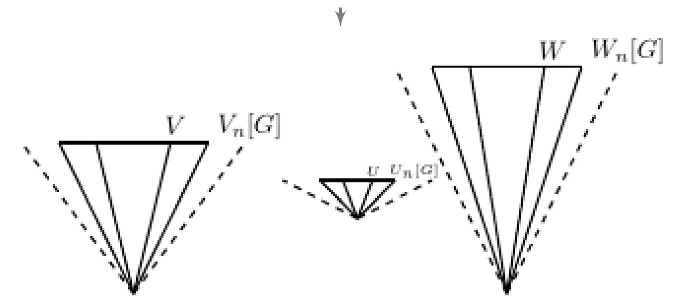

Figure 4: A representation of the Radical Multiverse

Set-generic multiverse: This multiverse is axiomatized and it is the collection of all extensions of V produced by *set-generic forcing*. It validates ZFC+ Large Cardinals.

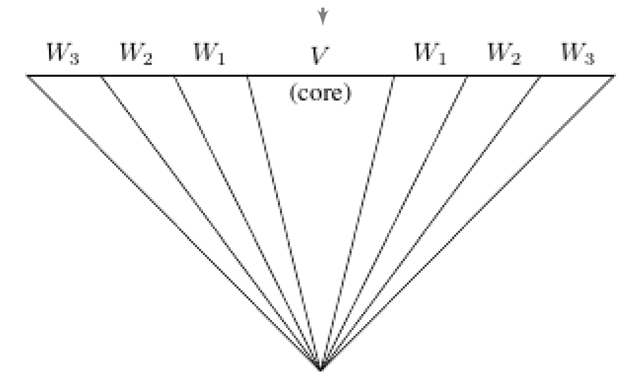

Figure 5: The set generic multiverse

***V*-logic Multiverse**: This multiverse uses the infinitary V-logic to define a multiverse of extensions of an *uncountable* V. It is axiomatized.

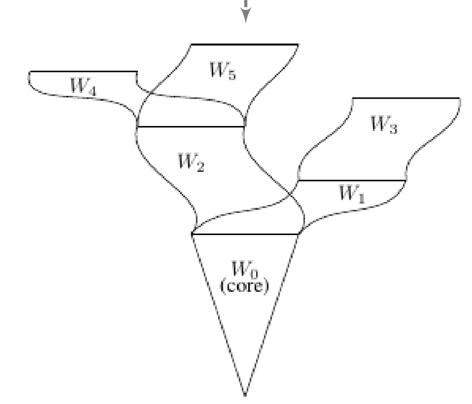

Figure 6: The V-logic multiverse

Hyperuniverse: Developed by Friedman, this multiverse is the collection of all countable transitive models of ZFC, starting from a *countable* V.

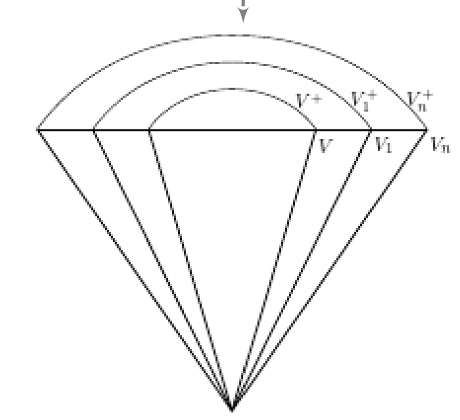

Figure 7: The Hyperuniverse

Parallel multiverse: The collection of all universes arising from the variation of the power set operation. This multiverse was developed by Väänänem, and it also has a metamatematics and logic.

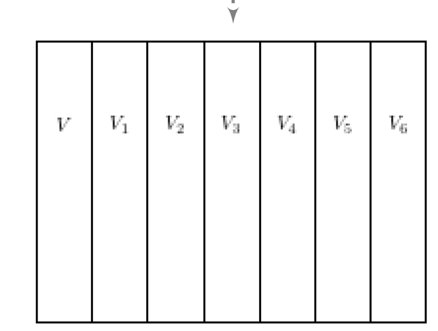

Figure 8: The Parallel universe